C24 Dynamical Systems

Lecture 4: Lyapunov analysis

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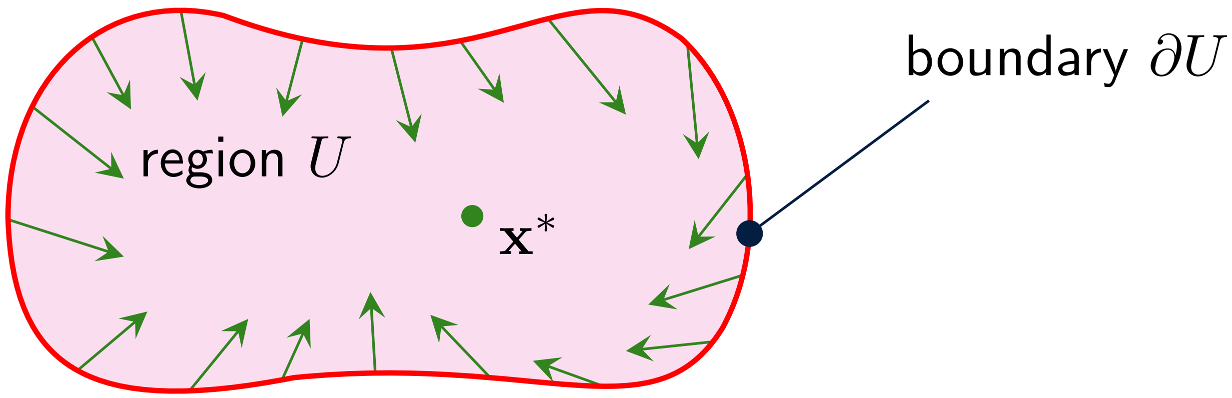
Lecture 4 overview

- This lecture focuses on **Lyapunov theory**, which can be used to check stability
- Lyapunov theory generalizes mechanical analyses that examine how energy is retained or lost in a system over time
- We will develop general principles of the theory – particularly the concept of a **Lyapunov function** – and use it to prove stability
- We will see how the method applies to **Hamiltonian systems,** which are mechanical systems that conserve total energy
- The related idea of a **gradient system** will be discussed

Lyapunov analysis: flow across a boundary

- We can get information about stability by thinking of how flows cross the boundary of a region U in phase space

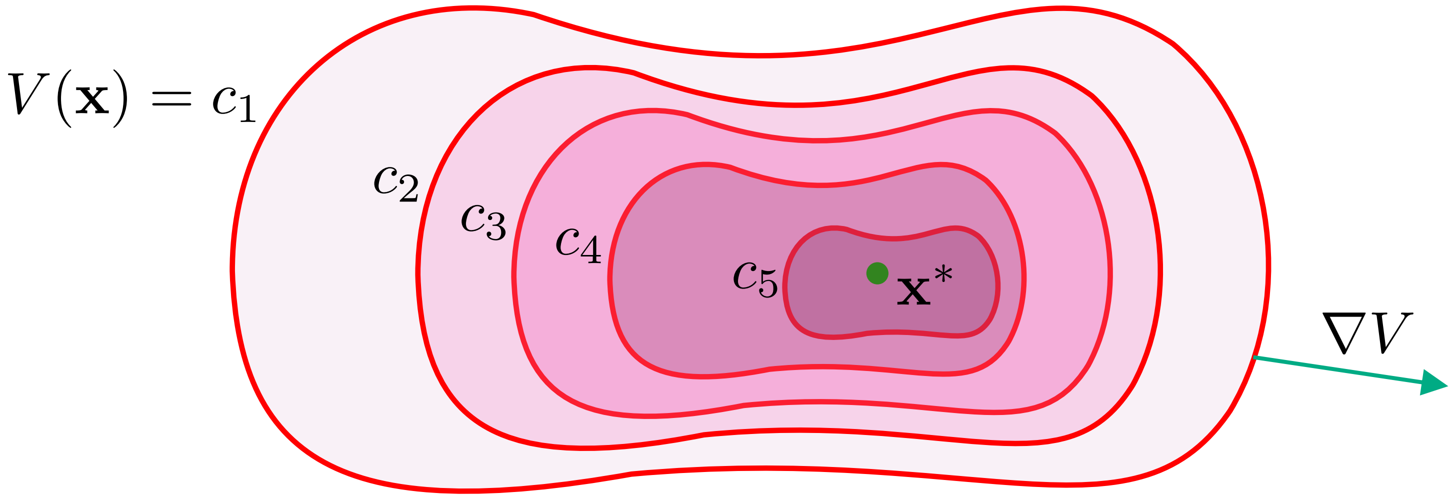


- Suppose we have a region U around an equilibrium point; if the vector field representing the flow of trajectories at the boundary ∂U always points inward or is tangential, then the flow can't escape

- We imagine this boundary to be drawn by a scalar function V, such that $V(\mathbf{x}) = c =$ constant

Lyapunov analysis: nested boundaries

- Now imagine that we have a nested set of boundary surfaces, described by ever smaller values of c in $V(\mathbf{x}) = c$



- If all the flow points inward, then $\dot{V} = \nabla V \cdot \dot{\mathbf{x}} \leq 0$ on the boundary
  - but $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, so we can instead write $\dot{V} = \nabla V \cdot \mathbf{f} \leq 0$
  - and if $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$, then V converges to a minimum point

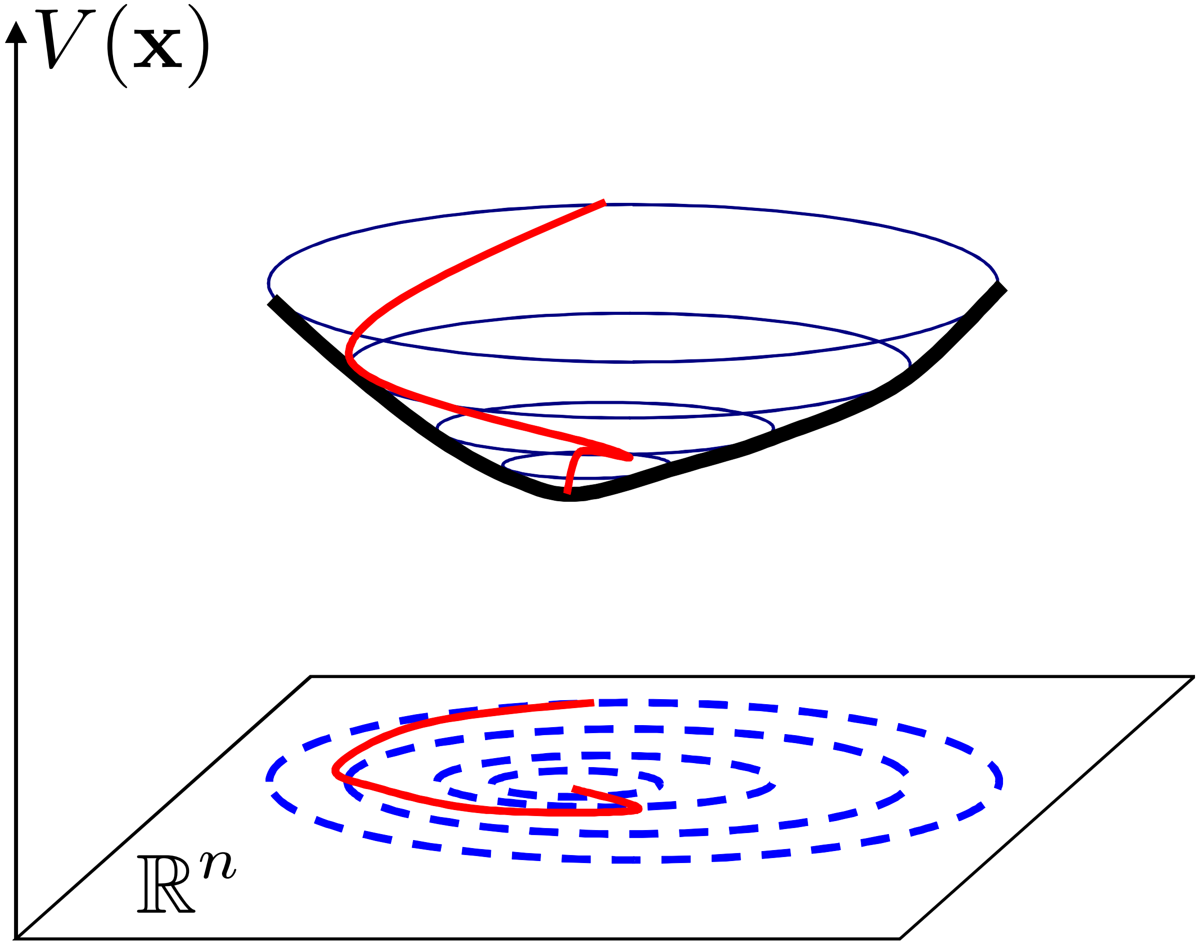
Lyapunov stability theorem

- Let $\mathbf{x}^*$ be an equilibrium point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ i.e. $\mathbf{f}(\mathbf{x}^*) = 0$
  Let D be an open set surrounding $\mathbf{x}^*$ and let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

  1. $V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$
  2. $\dot{V}(\mathbf{x}) = \nabla V \cdot \mathbf{f}(\mathbf{x}) \leq 0$

  then the equilibrium point $\mathbf{x}^*$ is **stable**

Illustration



The Lyapunov function $V(\mathbf{x})$ is non-increasing along solution trajectories

Example 1

Consider the nonlinear autonomous system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + \varepsilon x^2 y\end{aligned}$$

- Single equilibrium point at $(x^*, y^*) = (0, 0)$, with Jacobian

$$D\mathbf{f}(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \leftarrow$$

eigenvalues are $\lambda = \pm j$ (non-hyperbolic so Hartman-Grobman doesn't apply)

- Let the Lyapunov function be $V(x, y) = \frac{1}{2}(x^2 + y^2)$

$$\frac{dV}{dt} = \nabla V \cdot \dot{\mathbf{x}} = x\dot{x} + y\dot{y} = xy - xy + \varepsilon x^2 y^2 = \varepsilon x^2 y^2$$

so equilibrium $(0, 0)$ is **stable** if $\varepsilon \leq 0$

Example 2

Consider another autonomous system

$$\begin{aligned}\dot{x}_1 &= -2x_2 + x_2x_3\\ \dot{x}_2 &= x_1 - x_1x_3\\ \dot{x}_3 &= x_1x_2\end{aligned}$$

- Equilibrium point $(x_1, x_2, x_3) = (0, 0, 0)$ is a linear centre
- Define a Lyapunov function $V(\mathbf{x}) = c_1x_1^2 + c_2x_2^2 + c_3x_3^2$

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \frac{\partial V}{\partial x_3}\dot{x}_3\\ &= 2(c_2 - 2c_1)x_1x_2 + 2(c_1 - c_2 + c_3)x_1x_2x_3\end{aligned}$$

- Choose $c_2 = 2c_1 = c$, $c_3 = c_1 = \frac{1}{2}c$ and $c > 0$
  then $V > 0$ whenever $\mathbf{x} \neq 0$ and $\dot{V} = 0$ so $\mathbf{x} = 0$ is **stable**

Example 3: Jet engine

A simple model of a jet engine with a controller is

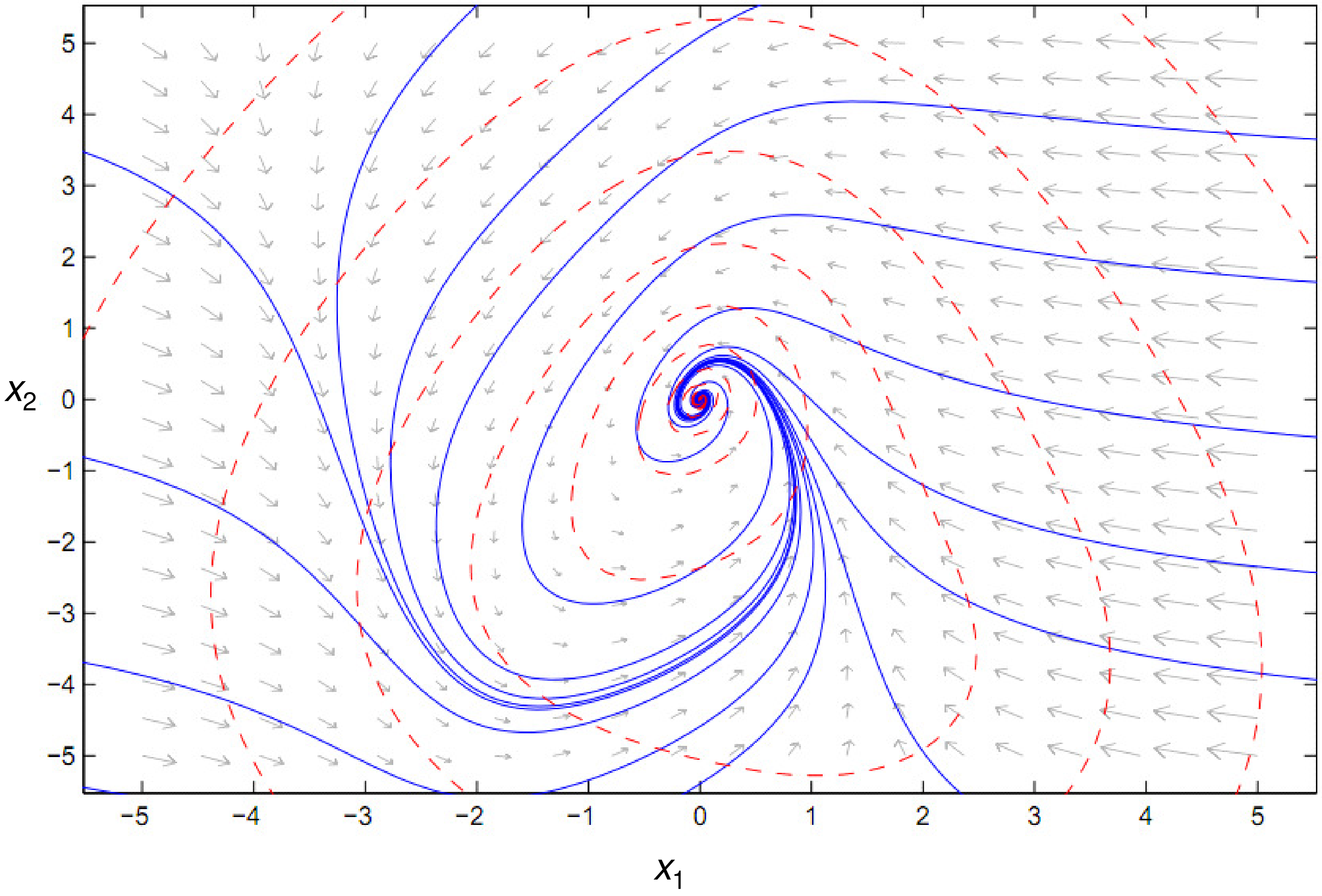
$$\dot{x}_1 = -x_2 + 1.5x_1^2 - 0.5x_1^3$$
$$\dot{x}_2 = 3x_1 - x_2$$

- Equilibrium at $(0,0)$ has $D\mathbf{f}(0,0) = \begin{bmatrix} 0 & -1 \\ 3 & -1 \end{bmatrix} \Rightarrow \lambda = \frac{-1 \pm j\sqrt{11}}{2}$
- The linearized system has a stable focus, so Hartman–Grobman says the system is stable near the origin (but not how near)
- Lyapunov functions can extend this result to prove global stability
- The function is quartic (plot on next slide):

$$V(\mathbf{x}) = c_1 x_1^2 + c_2 x_2^2 + c_3 x_1 x_2 + c_4 x_1^3 + \ldots + c_k x_2^4$$

Example 3: Jet engine

Solution trajectories in the phase plane



Dotted lines are the contours of the Lyapunov function

Vector fields possessing an integral

- Lyapunov functions can be cast in more intuitive terms by thinking of a physical system described by a potential

- The solution trajectories (or flow) of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a vector field. This vector field is said to have an **integral** $I(\mathbf{x})$ if

$$\frac{d}{dt}I(\mathbf{x}) = \sum_k \frac{\partial I}{\partial x_k}\dot{x}_k = \nabla I \cdot \dot{\mathbf{x}} = \nabla I \cdot \mathbf{f} = 0$$

- Here ∇I is the **gradient vector** of I

- The scalar function $I(\mathbf{x})$ defines **level sets** that contain the flow

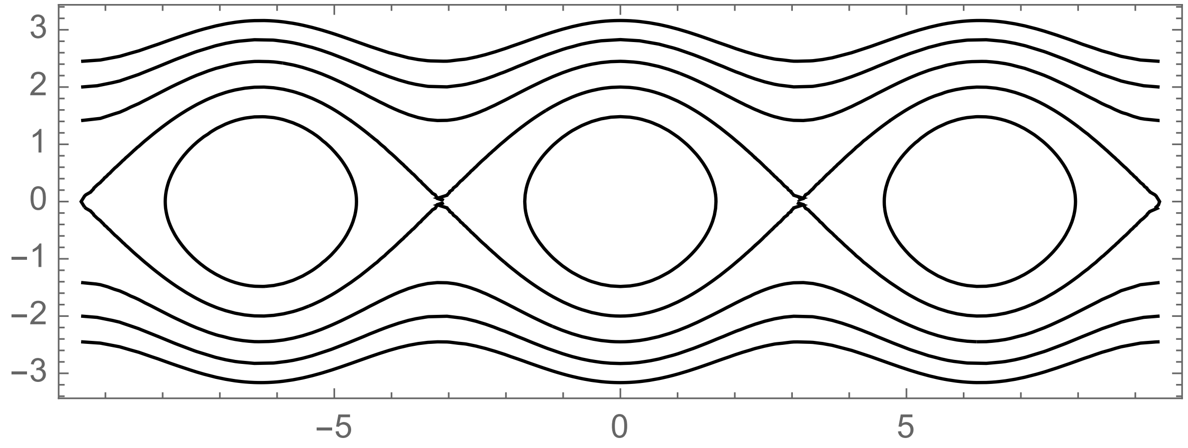
Example: Simple pendulum

- Governing system: $$\frac{d\theta}{dt} = p$$ $$\frac{dp}{dt} = -\frac{g}{\ell}\sin\theta$$

- The total stored energy is conserved, $E = \frac{1}{2}p^2 - \frac{g}{\ell}\cos\theta$

- This is consistent since $\frac{dE}{dt} = p\frac{dp}{dt} + \frac{g}{\ell}\sin\theta\frac{d\theta}{dt} = 0$

- Level sets of energy in state space:



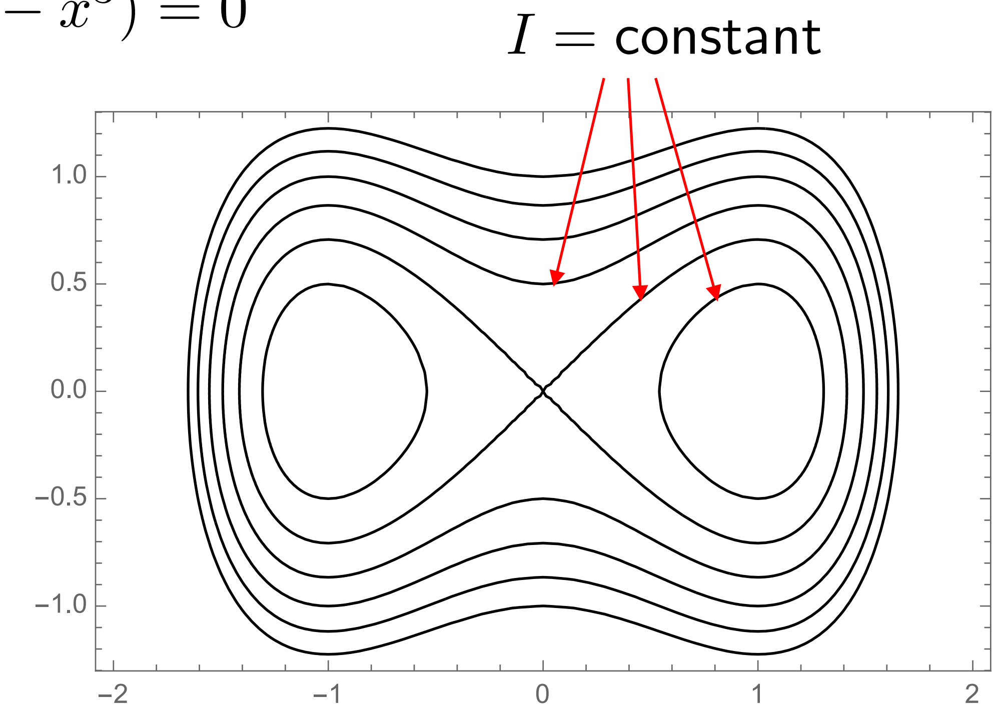
Example: Undamped Duffing oscillator

- Governing system (for $\gamma = 0$): $$\dot{x} = y$$ $$\dot{y} = x - x^3$$

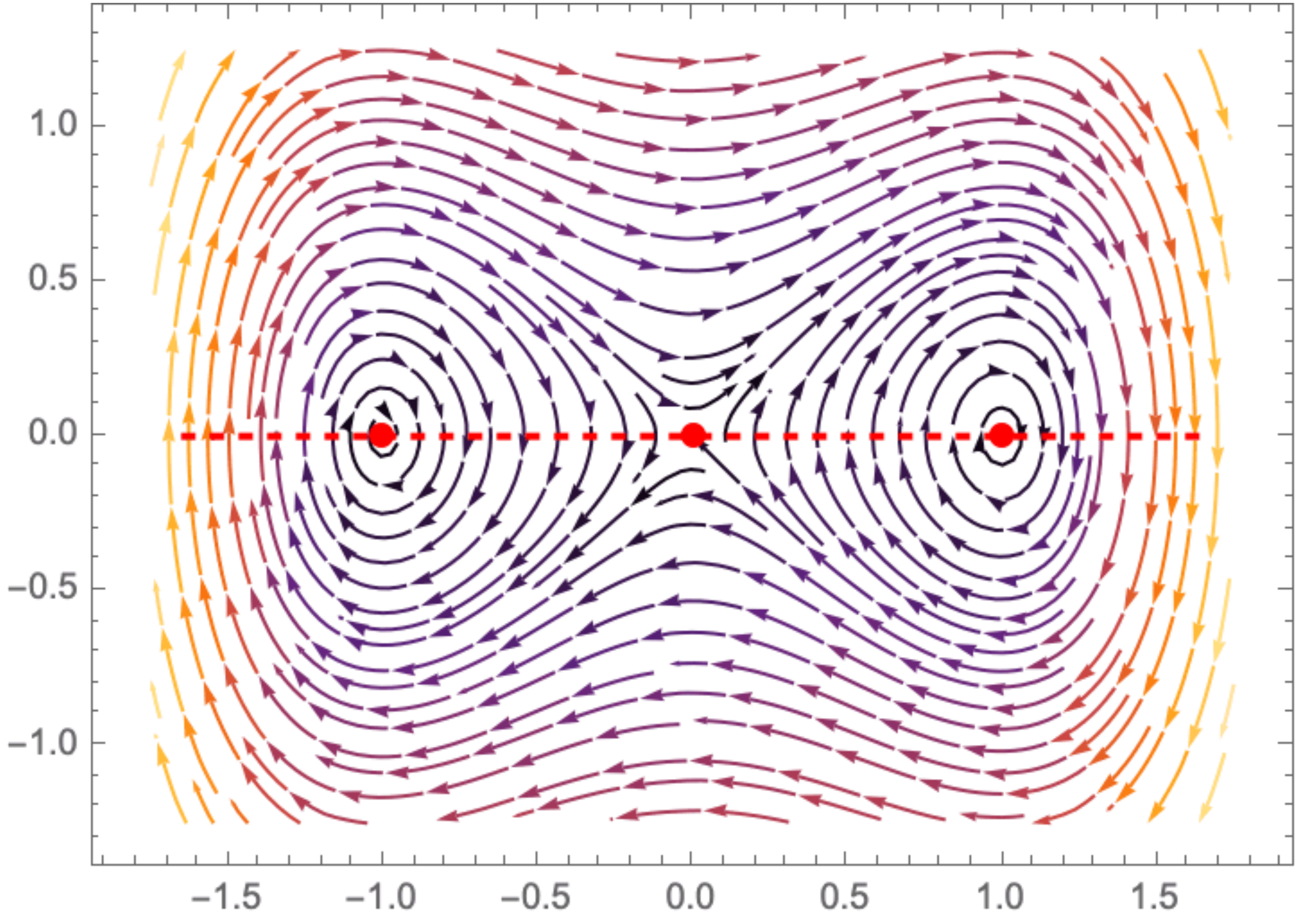
- Integral $I(x, y)$ satisfies $$\frac{dI}{dt} = \frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} = \frac{\partial I}{\partial x} y + \frac{\partial I}{\partial y}(x - x^3) = 0$$

- Separate variables and integrate:
$$\int y\,dy = \int (x - x^3)\,dx$$
$$\Rightarrow \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 + c$$

- Let $I = c$, then $I = \frac{1}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{2}y^2$



Example: Undamped Duffing oscillator



Hamiltonian systems

- **Hamilton's equations** provide an alternative way of phrasing Newton's laws – useful for conservative many-body systems
- The **Hamiltonian function** H casts the total energy (kinetic $+$ potential) of in terms of particle positions $\mathbf{q}$ and momenta $\mathbf{p}$
- Given $H(\mathbf{p}, \mathbf{q})$, a **Hamiltonian system** is defined as

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{f}(\mathbf{p}, \mathbf{q}) \\ \dot{\mathbf{q}} &= \mathbf{g}(\mathbf{p}, \mathbf{q}) \end{aligned} \qquad \text{where} \qquad \begin{aligned} f_i(\mathbf{p}, \mathbf{q}) &= -\frac{\partial H}{\partial q_i} \\ g_j(\mathbf{p}, \mathbf{q}) &= \frac{\partial H}{\partial p_j} \end{aligned}$$

- Here $\mathbf{p}$ and $\mathbf{q}$ are vectors with equal numbers n of real entries

Some facts about Hamiltonian systems

- If an equilibrium point $(\mathbf{p}^*, \mathbf{q}^*)$ is a (possibly local) minimum point of $H(\mathbf{p}, \mathbf{q})$, then it is a **stable** equilibrium point

- A Newtonian system of the form

$$\frac{d^2x}{dt^2} = f(x)$$

can be written as a Hamiltonian system by defining the Hamiltonian function as {potential energy} + {kinetic energy}:

$$H(x, v) = \frac{v^2}{2} - \int_{x_0}^{x} f(x)\, dx \quad \Longrightarrow \quad \begin{cases} \dfrac{\partial H}{\partial v} = v & \Longrightarrow \quad \dfrac{dx}{dt} = v \\ -\dfrac{\partial H}{\partial x} = f(x) & \Longrightarrow \quad \dfrac{dv}{dt} = f(x) \end{cases}$$

Gradient systems

Definition: A system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is referred to as a **gradient system** if there is a twice differentiable function $V(\mathbf{x})$ such that

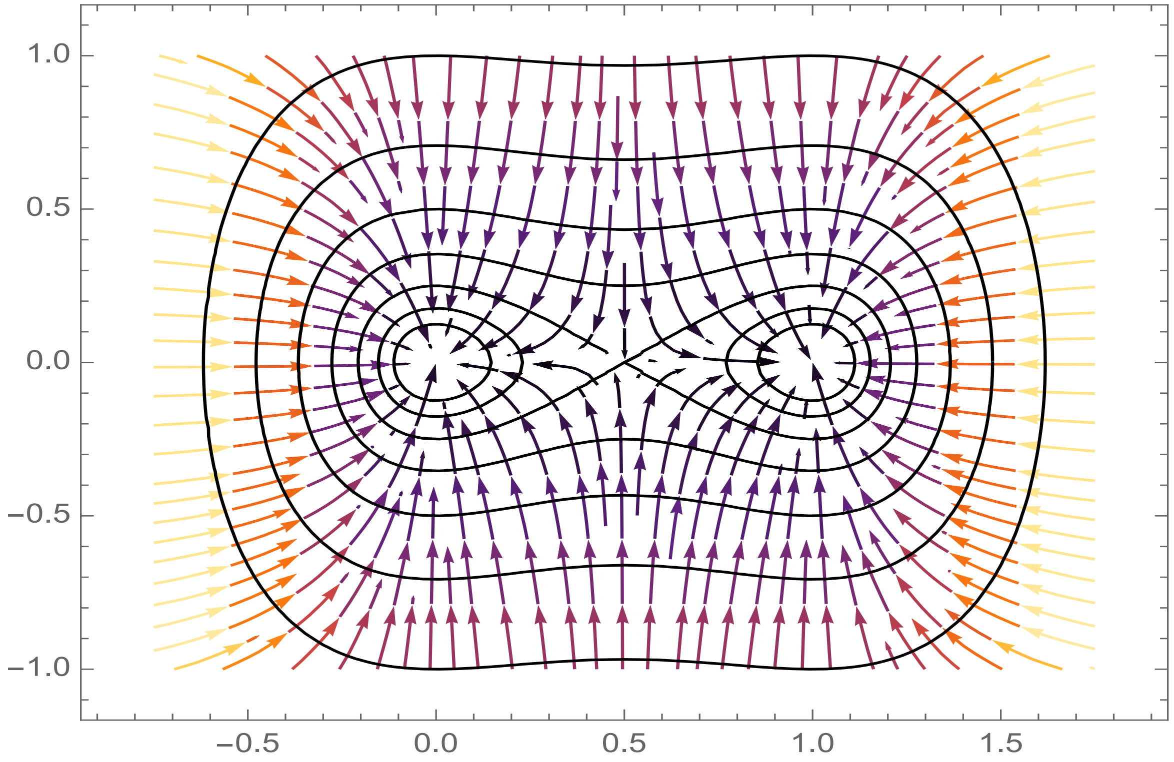
$$\frac{dx_i}{dt} = -\frac{\partial V}{\partial x_i} \quad \text{or} \quad f_i(\mathbf{x}) = -\frac{\partial V}{\partial x_i}$$

- Generally, equilibrium points are the **critical points** of V
- Away from equilibria, solution trajectories are orthogonal to the level sets of V (i.e. contours or surfaces of constant V)
- If $\mathbf{x}^*$ is a strict local **minimum** of V, then $V(\mathbf{x}) - V(\mathbf{x}^*)$ is a Lyapunov function showing that $\mathbf{x}^*$ is **asymptotically stable**
- If $\mathbf{x}^*$ is a strict local **maximum** of V, then $\mathbf{x}^*$ is **unstable**.

Example: Gradient system

The system
$$\dot{x} = -4x(x-1)(x-0.5)$$
$$\dot{y} = -2y$$

has potential
$$V(x,y) = \int 4x(x-1)(x-0.5)\,dx + \int 2y\,dy = x^2(x-1)^2 + y^2$$



Connecting gradient and Hamiltonian systems

- Consider the 2nd order Hamiltonian system

$$\dot{x} = f(x, y) = \frac{\partial H}{\partial y}$$

$$\dot{y} = g(x, y) = -\frac{\partial H}{\partial x}$$

- The solution flows of this system are orthogonal to the solution flows of the 2nd order gradient system

$$\dot{x} = g(x, y) = -\frac{\partial H}{\partial x}$$

$$\dot{y} = -f(x, y) = -\frac{\partial H}{\partial y}$$

- These two systems have the same equilibria; centres map to nodes (real λ with same sign); saddles map to saddles, and foci of the flows map to foci

Questions?